Minimum Sum Euclidean Decompositions of Integers

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1 Objective

Let n denote a positive integer. Throughout, $\lfloor n \rfloor$ will denote the largest integer function evaluated at the number n. We define a **Euclidean Decomposition** of n as a triple (x, y, z) of non-negative integers satisfying $n = x \cdot y + z$, $0 < x,\ y \leq n$, $0 \leq z < x$. We call the numbers x, y and z the **terms of the decomposition**. For fixed $0 < x \leq n$, it is easily verified that the equation has the unique solution $y = \left\lfloor \frac{n}{x} \right\rfloor$, $z = n - x \cdot y$.

One may then ask how one finds a Euclidean decomposition (x, y, z) of n with the minimum sum $x + y + z$ of its constituent terms. It is easy to see that such a decomposition can always be found in time $\mathcal{O}(n^{\frac{1}{2}})$, by trying all choices for x between 1 and $\lfloor \sqrt{n} \rfloor + 2$. We will show in this work that there also exists an algorithm with complexity $\mathcal{O}(n^{\frac{3}{8}})$ for this purpose.

2 Preliminaries

Lemma 1.

1. *$n \geq \lfloor \sqrt{n} \rfloor^2$ with equality if and only if n is a perfect square.*
2. *$n \leq \lfloor \sqrt{n} \rfloor \cdot (\lfloor \sqrt{n} \rfloor + 2)$*

Proof.

1. This follows from the fact that $\lfloor \sqrt{n} \rfloor \leq \sqrt{n}$ with equality if and only if $\sqrt{n}$ is an integer, i.e. n is a perfect square.
2. Note that $\lfloor \sqrt{n} \rfloor > \sqrt{n} - 1$ by the definition of the floor function. We have

$$\begin{aligned}
\lfloor \sqrt{n} \rfloor \cdot \left(\lfloor \sqrt{n} \rfloor + 2\right) &= \lfloor \sqrt{n} \rfloor^2 + 2 \cdot \lfloor \sqrt{n} \rfloor \\
&> \left(\sqrt{n} - 1\right)^2 + 2 \cdot \left(\sqrt{n} - 1\right) \\
&= \left(n + 1 - 2 \cdot \sqrt{n}\right) + 2 \cdot \sqrt{n} - 2 \\
&= n - 1
\end{aligned}$$

   Since $\lfloor \sqrt{n} \rfloor \cdot (\lfloor \sqrt{n} \rfloor + 2)$ and n are both integers, the above inequality implies that $\lfloor \sqrt{n} \rfloor \cdot (\lfloor \sqrt{n} \rfloor + 2) \geq n$, as required.

□

Remark 1. Let $x \cdot y = n = x' \cdot y'$, with $x = \lfloor\sqrt{n}\rfloor - t_1,\ y = \lfloor\sqrt{n}\rfloor + s_1,\ x' = \lfloor\sqrt{n}\rfloor - t_2,\ y' = \lfloor\sqrt{n}\rfloor + s_2$. Then, $t_1 < t_2 \iff s_1 < s_2$.

Lemma 2. *Let*

$$n = x \cdot y + z, \quad 1 \le x \le y < n, \quad 0 \le z < x \tag{1}$$

Then, without loss of generality we can write

$$x = \lfloor\sqrt{n}\rfloor - t > 0, \qquad y = \lfloor\sqrt{n}\rfloor + s > 0, \qquad \text{with } 0 \le t \le s \tag{2}$$

Consequently,

$$x + y + z = 2 \cdot \lfloor\sqrt{n}\rfloor + (s - t) + z; \ \ s \ge t \ge 0 \tag{3}$$

So, $x + y + z \ge 2 \cdot \lfloor\sqrt{n}\rfloor$. Moreover, we have $z \ge t^2$.

Proof. First note that whenever $n = x \cdot y + z$, with $z < x \le y$, we have $\left\lfloor \frac{n}{x} \right\rfloor = y$. Since the equation is symmetric in x and y, it is enough to consider the case $x < y$. For any $s \ge 0$, we have

$$\lfloor\sqrt{n}\rfloor \ge \left\lfloor \frac{n}{\lfloor\sqrt{n}\rfloor + s} \right\rfloor \ge \lfloor\sqrt{n}\rfloor - s$$

Thus, we necessarily have $x \le \lfloor\sqrt{n}\rfloor$, or in other words x must be of the form $(\lfloor\sqrt{n}\rfloor - t)\,,\ \lfloor\sqrt{n}\rfloor > t \ge 0$. For any such t,

$$\begin{aligned} \frac{n}{\lfloor\sqrt{n}\rfloor - t} \ge \frac{n}{\sqrt{n} - t} \ge \frac{n}{n - t^2} \cdot \left(\sqrt{n} + t\right) \\ \implies y = \left\lfloor \frac{n}{\lfloor\sqrt{n}\rfloor - t} \right\rfloor \ge \lfloor\sqrt{n}\rfloor + t \end{aligned}$$

Thus, y must be of the form $\lfloor\sqrt{n}\rfloor + s$, with $s \ge t \ge 0$. So, $x + y$, and thus $x + y + z$, is greater than or equal to $2 \cdot \lfloor\sqrt{n}\rfloor$. Moreover, we have

$$\begin{aligned} z &= n - (\lfloor\sqrt{n}\rfloor - t)(\lfloor\sqrt{n}\rfloor + s) \\ &\ge n - (\lfloor\sqrt{n}\rfloor - t)(\lfloor\sqrt{n}\rfloor + t) \\ &\ge t^2 \end{aligned} \tag{4}$$

□

Proposition 1. *For any $m \ge 0$, the sequence $x_r = \{(x-r)(x+r+m)\}_{r \ge 0}$ is strictly decreasing in r.*

Proof. We have, for $r > 0$,

$$\begin{aligned} x_r - x_{r+1} &= (x - r) \cdot (x + r + m) - (x - r - 1) \cdot (x + r + m + 1) \\ &= m + 1 + 2 \cdot r > 0 \end{aligned}$$

□

3 Finding the minimum sum decpmposition

3.1 CASE $\lfloor\sqrt{n}\rfloor$ divides n

Theorem 1. *Let $n \geq 1$, and assume that $\lfloor\sqrt{n}\rfloor$ divides n. The tuple $(p,q,r) = \left(\lfloor\sqrt{n}\rfloor, \left\lfloor \frac{n}{\lfloor\sqrt{n}\rfloor} \right\rfloor, 0\right)$ satisfies*

$$p + q + r = \min\left(\{(x,y,z) \mid (x,y,z) \text{ is a solution of equation (1) }\}\right)$$

Proof. We have, using Lemma 1,

$$\begin{aligned} \left\lfloor \frac{n}{\lfloor\sqrt{n}\rfloor} \right\rfloor = \frac{n}{\lfloor\sqrt{n}\rfloor} &= \lfloor\sqrt{n}\rfloor \\ \text{or } &\lfloor\sqrt{n}\rfloor + 1 \\ \text{or } &\lfloor\sqrt{n}\rfloor + 2 \end{aligned}$$

We now examine each sub-case separately.

1. CASE $n = \lfloor\sqrt{n}\rfloor^2$

   Here, we have $n = \lfloor\sqrt{n}\rfloor^2$, so n is a perfect square, so $\sqrt{n} = \lfloor\sqrt{n}\rfloor$, $z = s = t = 0$, and, by equation (3), the minimum possible sum has value $2 \cdot \lfloor\sqrt{n}\rfloor = 2 \cdot \sqrt{n}$. The proof is complete for this case.

2. CASE $n = \lfloor\sqrt{n}\rfloor \cdot (\lfloor\sqrt{n}\rfloor + 1)$

   We have, for any $t < \lfloor\sqrt{n}\rfloor$,

$$\begin{aligned} s &= \frac{\lfloor\sqrt{n}\rfloor \cdot (\lfloor\sqrt{n}\rfloor + 1)}{\lfloor\sqrt{n}\rfloor - t} - \lfloor\sqrt{n}\rfloor \\ &= \lfloor\sqrt{n}\rfloor + 1 + \frac{\lfloor\sqrt{n}\rfloor + 1}{\lfloor\sqrt{n}\rfloor - t} \cdot t - \lfloor\sqrt{n}\rfloor \\ &\geq t + 1 \end{aligned}$$

   If $t \geq 1$, $x + y + z = 2\lfloor\sqrt{n}\rfloor + (s - t) + z \geq 2\lfloor\sqrt{n}\rfloor + (s - t) + t^2 \geq 2\lfloor\sqrt{n}\rfloor + 2$.

   If $t = 0$, $s = \left\lfloor \frac{n}{\lfloor\sqrt{n}\rfloor - t} \right\rfloor - \lfloor\sqrt{n}\rfloor = 1$ and $z = 0$, and thus the sum is equal to $2 \cdot \lfloor\sqrt{n}\rfloor + 1$. By (3), $p = \lfloor\sqrt{n}\rfloor$ is as required.

3. CASE $n = \lfloor\sqrt{n}\rfloor \cdot (\lfloor\sqrt{n}\rfloor + 2)$

   We have, as before, for any $t < \lfloor\sqrt{n}\rfloor$,

$$
\begin{aligned}
s &= \frac{\lfloor\sqrt{n}\rfloor \cdot (\lfloor\sqrt{n}\rfloor + 2)}{\lfloor\sqrt{n}\rfloor - t} - \lfloor\sqrt{n}\rfloor \\
&= \lfloor\sqrt{n}\rfloor + 2 + \frac{(\lfloor\sqrt{n}\rfloor + 2)}{\lfloor\sqrt{n}\rfloor - t} \cdot t - \lfloor\sqrt{n}\rfloor \geq t + 2
\end{aligned}
$$

So, if $t \geq 1$, $x + y + z = 2\lfloor\sqrt{n}\rfloor + (s-t) + z \geq \lfloor\sqrt{n}\rfloor + (s-t) + t^2 + s - t \geq \lfloor\sqrt{n}\rfloor + 3$. If $t = 0$, we have $s = \left\lfloor \frac{n}{\lfloor\sqrt{n}\rfloor - t} \right\rfloor - \lfloor\sqrt{n}\rfloor = 2$, and $z = 0$. Thus, $x + y + z = 2 \cdot \lfloor\sqrt{n}\rfloor + 2$. By (3), this is the minimum sum attainable in this case, therefore $p = \lfloor\sqrt{n}\rfloor$.

□

3.2 CASE $\lfloor\sqrt{n}\rfloor$ does not divide n

Lemma 3. *Suppose that $n < \lfloor\sqrt{n}\rfloor \cdot (\lfloor\sqrt{n}\rfloor + 1)$. Then there exists $0 \leq a_0 < n^{\frac{1}{4}}$ such that*

$$
\left(\lfloor\sqrt{n}\rfloor - a_0 - 1\right) \cdot \left(\lfloor\sqrt{n}\rfloor + a_0 + 2\right) \leq n < \left(\lfloor\sqrt{n}\rfloor - a_0\right)\left(\lfloor\sqrt{n}\rfloor + a_0 + 1\right) \tag{5}
$$

On the other hand, if $n \geq \lfloor\sqrt{n}\rfloor \cdot (\lfloor\sqrt{n}\rfloor + 1)$, there exists $0 \leq r_0 < n^{\frac{1}{4}}$ such that

$$
\left(\lfloor\sqrt{n}\rfloor - r_0 - 1\right) \cdot \left(\lfloor\sqrt{n}\rfloor + r_0 + 3\right) \leq n < \left(\lfloor\sqrt{n}\rfloor - a_0\right)\left(\lfloor\sqrt{n}\rfloor + a_0 + 2\right) \tag{6}
$$

Proof. First suppose that $n < \lfloor\sqrt{n}\rfloor \cdot (\lfloor\sqrt{n}\rfloor + 1)$. By setting $m = 0$ in Proposition 1, we have a strictly decreasing integer sequence

$$
\left\{X_a = \left(\lfloor\sqrt{n}\rfloor - a\right)\left(\lfloor\sqrt{n}\rfloor + a + 1\right)\right\}_{a \geq 0}
$$

with first (and maximum) term equal to $\lfloor\sqrt{n}\rfloor \cdot (\lfloor\sqrt{n}\rfloor + 1)$. Also, for $a = \lfloor\sqrt{n}\rfloor$, $X_a = 0$. So, $X_{\lfloor\sqrt{n}\rfloor} < n < X_0$. Thus, there exists a_0 such that (3) is satisfied. Now, for any such a_0,

$$
\begin{aligned}
& n < \left(\lfloor\sqrt{n}\rfloor - a_0\right) \cdot \left(\lfloor\sqrt{n}\rfloor + a_0 + 1\right) < \lfloor\sqrt{n}\rfloor^2 - a_0^2 + \lfloor\sqrt{n}\rfloor - a_0 \\
\Longrightarrow\ & a_0^2 < a_0^2 + a_0 < \lfloor\sqrt{n}\rfloor \cdot \left(\lfloor\sqrt{n}\rfloor + 1\right) - n < \lfloor\sqrt{n}\rfloor \leq \sqrt{n}.
\end{aligned}
$$

Thus, $a_0 < n^{\frac{1}{4}}$, as required.

Now suppose that $n \geq \lfloor\sqrt{n}\rfloor \cdot (\lfloor\sqrt{n}\rfloor + 1)$. We know, by Lemma 1, that $n \leq \lfloor\sqrt{n}\rfloor \cdot (\lfloor\sqrt{n}\rfloor + 2)$. By setting $m = 2$ in Proposition 1, we have a strictly decreasing integer sequence

$$
\left\{Y_r = \left(\lfloor\sqrt{n}\rfloor - r\right)\left(\lfloor\sqrt{n}\rfloor + r + 2\right)\right\}_{a \geq 0}
$$

with first (and maximum) term equal to

$$
\lfloor\sqrt{n}\rfloor \cdot \left(\lfloor\sqrt{n}\rfloor + 2\right).
$$

Also, for $r = \lfloor\sqrt{n}\rfloor$, $Y_r = 0$. So, $Y_{\lfloor\sqrt{n}\rfloor} < n < Y_0$. Thus, there exists r_0 such that (6) is satisfied. Now, for any such r_0,

$$\begin{aligned} & n < \left(\lfloor\sqrt{n}\rfloor - r_0\right) \cdot \left(\lfloor\sqrt{n}\rfloor + r_0 + 2\right) < \lfloor\sqrt{n}\rfloor^2 - r_0^2 + 2\lfloor\sqrt{n}\rfloor - 2r_0 \\ \implies {r_0}^2 < {r_0}^2 + 2r_0 & < \lfloor\sqrt{n}\rfloor \cdot \left(\lfloor\sqrt{n}\rfloor + 1\right) - n + \lfloor\sqrt{n}\rfloor \\ & < \lfloor\sqrt{n}\rfloor \qquad \left(\text{since } n > \lfloor\sqrt{n}\rfloor \cdot \left(\lfloor\sqrt{n}\rfloor + 1\right)\right). \end{aligned}$$

Thus, $r_0 < n^{\frac{1}{4}}$, as required. □

Lemma 4. *Suppose that B is an upper bound for the quantity $(s - t)$. Then, we have*

$$t \le n^{\frac{1}{4}} \cdot (B+1)^{1/2}$$

Proof. We have,

$$\begin{aligned} s - t &= \left\lfloor \frac{n}{\lfloor\sqrt{n}\rfloor - t} \right\rfloor - \lfloor\sqrt{n}\rfloor - t > \frac{n}{\lfloor\sqrt{n}\rfloor - t} - \left(\lfloor\sqrt{n}\rfloor + t\right) - 1 = \frac{n - \lfloor\sqrt{n}\rfloor^2 + t^2}{\lfloor\sqrt{n}\rfloor - t} - 1 \\ \implies & \frac{t^2}{\lfloor\sqrt{n}\rfloor - t} < (s-t) + 1 \le B + 1 \\ \implies & t^2 \le (B+1) \cdot \lfloor\sqrt{n}\rfloor \le \sqrt{n} \cdot (B+1) \\ \implies & t \le n^{\frac{1}{4}} \cdot (B+1)^{1/2}. \end{aligned}$$

□

Proposition 2. *Suppose that $n < \lfloor\sqrt{n}\rfloor \cdot (\lfloor\sqrt{n}\rfloor + 1)$ and let a_0 be as in Lemma 3, and s and t be as in (2). Write*

$$x_0 = \lfloor\sqrt{n}\rfloor + a_0 + 2, \qquad y_0 = \lfloor\sqrt{n}\rfloor - a_0 - 1, \qquad z_0 = n - x_0 \cdot y_0 \tag{7}$$

$$x_1 = \lfloor\sqrt{n}\rfloor - t, \qquad y_1 = \lfloor\sqrt{n}\rfloor + s, \qquad z_1 = n - x_1 \cdot y_1 \tag{8}$$

Then, if $x_1 + y_1 + z_1 < x_0 + y_0 + z_0$, then $t < n^{\frac{1}{4}} \cdot (2n^{\frac{1}{4}} + 3)^{1/2}$.

Proof. First note that we have

$$\begin{aligned} & z_0 = n - x_0 \cdot y_0 = n - \left(\lfloor\sqrt{n}\rfloor - a_0 - 1\right)\left(\lfloor\sqrt{n}\rfloor + a_0 + 2\right) \\ \implies & z_0 < \left(\lfloor\sqrt{n}\rfloor - a_0\right)\left(\lfloor\sqrt{n}\rfloor + a_0 + 1\right) - \left(\lfloor\sqrt{n}\rfloor - a_0 - 1\right)\left(\lfloor\sqrt{n}\rfloor + a_0 + 2\right) \\ \implies & z_0 < 2 \cdot (a_0 + 1) \end{aligned} \tag{9}$$

By assumption, $x_1 + y_1 + z_1 < x_0 + y_0 + z_0$, so

$$\begin{aligned} & 2 \cdot \lfloor\sqrt{n}\rfloor + (s-t) + z_1 < 2 \cdot \lfloor\sqrt{n}\rfloor + 1 + 2 \cdot (a_0 + 1) \\ \therefore\ & (s-t) \le (s-t) + z_1 \le 2 \cdot (a_0 + 1) \end{aligned} \tag{10}$$

Also recall from Lemma (3) that we have $a_0 < n^{\frac{1}{4}}$. Now, applying the upper bound from equation (9) to the claim above, we get

$$t \leq n^{\frac{1}{4}} \cdot (2a_0 + 3)^{1/2} < n^{\frac{1}{4}} \cdot (2n^{\frac{1}{4}} + 3)^{1/2}.$$

□

Proposition 3. *Suppose that* $n \geq \lfloor\sqrt{n}\rfloor \cdot (\lfloor\sqrt{n}\rfloor + 1)$ *and let* r_0 *be as in Lemma 3, and* s *and* t *be as in (2). Write*

$$x_0 = \left\lfloor\sqrt{n}\right\rfloor + r_0 + 3, \qquad y_0 = \left\lfloor\sqrt{n}\right\rfloor - r_0 - 1, \qquad z_0 = n - x_0 \cdot y_0 \tag{11}$$
$$x_1 = \left\lfloor\sqrt{n}\right\rfloor - t, \qquad y_1 = \left\lfloor\sqrt{n}\right\rfloor + s, \qquad z_1 = n - x_1 \cdot y_1 \tag{12}$$

Then, if $x_1 + y_1 + z_1 < x_0 + y_0 + z_0$*, then* $t < n^{\frac{1}{4}} \cdot (2n^{\frac{1}{4}} + 4)^{1/2}$.

Proof. First note that we have

$$\begin{aligned} z_0 = n - x_0 \cdot y_0 = n - \left(\left\lfloor\sqrt{n}\right\rfloor - r_0\right)\left(\left\lfloor\sqrt{n}\right\rfloor + r_0 + 2\right) \\ \implies z_0 < \left(\left\lfloor\sqrt{n}\right\rfloor - r_0\right)\left(\left\lfloor\sqrt{n}\right\rfloor + r_0 + 2\right) - \left(\left\lfloor\sqrt{n}\right\rfloor - r_0 - 1\right)\left(\left\lfloor\sqrt{n}\right\rfloor + r_0 + 3\right) \\ \implies z_0 < 2 \cdot r_0 + 3 \end{aligned} \tag{13}$$

By assumption, $x_1 + y_1 + z_1 < x_0 + y_0 + z_0$, so

$$\begin{aligned} 2 \cdot \left\lfloor\sqrt{n}\right\rfloor + (s - t) + z_1 < 2 \cdot \left\lfloor\sqrt{n}\right\rfloor + 1 + 2 \cdot r_0 + 3 \\ \therefore (s - t) \leq (s - t) + z_1 < 2 \cdot r_0 + 4 \end{aligned} \tag{14}$$

Also recall from Lemma 3 that we have $r_0 < n^{\frac{1}{4}}$. Now, applying the upper bound from Lemma 4 to the claim above, we get

$$t \leq n^{\frac{1}{4}} \cdot (2r_0 + 4)^{1/2} < n^{\frac{1}{4}} \cdot (2n^{\frac{1}{4}} + 4)^{1/2}.$$

□

We are now ready to state the algorithm which we show in Theorem 1 to find a Euclidean decomposition with minimum sum.

Algorithm 1: Sum Minimization

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if $n < \lfloor\sqrt{n}\rfloor \cdot (\lfloor\sqrt{n}\rfloor + 1)$ **then**

  $a \leftarrow 0.$

  while $n < (\lfloor\sqrt{n}\rfloor - a) \cdot (\lfloor\sqrt{n}\rfloor + a + 1)$ **do**

    $a \leftarrow a + 1.$

  $\alpha_1 \leftarrow \lfloor\sqrt{n}\rfloor - a,\ \ \beta_1 \leftarrow \lfloor\sqrt{n}\rfloor + a + 1,\ \ \gamma_1 \leftarrow n - \alpha_1 \cdot \beta_1.$

  $T \leftarrow n^{\frac{1}{4}} \cdot (2a+3)^{1/2}.$

  for $1 \leq t \leq T$ **do**

    1. Calculate $s = \left\lfloor \frac{n}{\lfloor\sqrt{n}\rfloor - t} \right\rfloor - \lfloor\sqrt{n}\rfloor$ and $z := n - (\lfloor\sqrt{n}\rfloor - t) \cdot (\lfloor\sqrt{n}\rfloor + s)$.
    2. **if** $2\lfloor\sqrt{n}\rfloor + s - t + z \leq \alpha_t + \beta_t + \gamma_t$ **then**

      $\alpha_{t+1} \leftarrow \lfloor\sqrt{n}\rfloor - t,\ \beta_{t+1} \leftarrow \lfloor\sqrt{n}\rfloor + s,\ \gamma_{t+1} \leftarrow z.$

      else

      $\alpha_{t+1} \leftarrow \alpha_t,\ \beta_{t+1} \leftarrow \beta_t,\ \gamma_{t+1} \leftarrow \gamma_t.$

else

  $r \leftarrow 0.$

  while $n < (\lfloor\sqrt{n}\rfloor - r) \cdot (\lfloor\sqrt{n}\rfloor + r + 2)$ **do**

    $r \leftarrow r + 1.$

  $\alpha_1 \leftarrow \lfloor\sqrt{n}\rfloor - r,\ \ \beta_1 \leftarrow \lfloor\sqrt{n}\rfloor + r + 1,\ \ \gamma_1 \leftarrow n - \alpha_1 \cdot \beta_1.$

  $T \leftarrow n^{\frac{1}{4}} \cdot (2r+4)^{1/2}.$

  for $1 \leq t \leq T$ **do**

    1. Calculate $s = \left\lfloor \frac{n}{\lfloor\sqrt{n}\rfloor - t} \right\rfloor - \lfloor\sqrt{n}\rfloor$ and $z := n - (\lfloor\sqrt{n}\rfloor - t) \cdot (\lfloor\sqrt{n}\rfloor + s)$.
    2. **if** $2\lfloor\sqrt{n}\rfloor + s - t + z \leq \alpha_t + \beta_t + \gamma_t$ **then**

      $\alpha_{t+1} \leftarrow \lfloor\sqrt{n}\rfloor - t,\ \beta_{t+1} \leftarrow \lfloor\sqrt{n}\rfloor + s,\ \gamma_{t+1} \leftarrow z.$

      else

      $\alpha_{t+1} \leftarrow \alpha_t,\ \beta_{t+1} \leftarrow \beta_t,\ \gamma_{t+1} \leftarrow \gamma_t.$

Return $(p,q,r) := (\alpha_T, \beta_T, \gamma_T)$.

Theorem 2. *Algorithm 1 terminates in $\mathcal{O}(n^{\frac{3}{8}})$ steps, and its output (p,q,r) of satisfies*

$$p + q + r = \min\left(\{(x,y,z) \mid (x,y,z) \text{ is a solution of equation } (1)\ \}\right)$$

Proof. First suppose that $n < \lfloor\sqrt{n}\rfloor \cdot (\lfloor\sqrt{n}\rfloor + 1)$. Let (x_1, y_1, z_1) be a solution of Equation (1) producing the minimum sum and let s and t be as in Equation (8). Also let x_0, y_0, and z_0 be as in (7). If the minimum possible sum is less than $x_0 + y_0 + z_0$, then by the proof of Proposition 2, we have $t \leq n^{\frac{1}{4}} \cdot (2a+3)^{1/2}$ if $n < \lfloor\sqrt{n}\rfloor \cdot (\lfloor\sqrt{n}\rfloor + 1)$, where r and a are the values as in Lemma 3, which are calculated by the algorithm. The algorithm goes through every such value of t and records each new tuple producing a smaller sum, returning the tuple giving the smallest sum, which is, by the argument above, the minimum. If the minimum sum equals $x_0 + y_0 + z_0$, then the algorithm by default returns the tuple (x_0, y_0, z_0). An analogous argument holds for the second part of the algorithm, which runs if $n \geq \lfloor \sqrt{n} \rfloor \cdot (\lfloor \sqrt{n} \rfloor + 1)$. Finally, note that the algorithm calculates a in $\mathcal{O}(n^{\frac{1}{4}})$ steps, by Lemma 3, and then performs $T = n^{\frac{1}{4}} \cdot (2n^{\frac{1}{4}} + 3)^{1/2} = \mathcal{O}(n^{\frac{3}{8}})$ more iterations, thus having a total complexity of $\mathcal{O}(n^{\frac{3}{8}})$. $\square$